J. Fluid Mech. (1967), *vol.* 28, *part* 3, *pp.* 593–616
Printed in Great Britain

Calculation of boundary-layer development using the turbulent energy equation

By P. BRADSHAW, D. H. FERRISS AND N. P. ATWELL†
Aerodynamics Division, National Physical Laboratory, Teddington

(Received 9 July 1966)

The turbulent energy equation is converted into a differential equation for the turbulent shear stress by defining three empirical functions relating the turbulent intensity, diffusion and dissipation to the shear stress profile. This equation, the mean momentum equation and the mean continuity equation form a hyperbolic system. Numerical integrations by the method of characteristics with preliminary choices of the three empirical functions compare favourably with the results of conventional calculation methods over a wide range of pressure gradients. Nearly all the empirical information required has been derived solely from the boundary layer in zero pressure gradient.

1. Introduction

The basic assumption made in most present-day methods of calculating the development of turbulent boundary layers is that the shear-stress profile at a given distance from the origin of the boundary layer is uniquely related to the mean-flow conditions at that station. The simplest version of this assumption is the 'mixing-length' or 'eddy viscosity' assumption that the shear stress at a point depends on the mean velocity gradient at that point: the empirical equations for dH/dx used in the integral methods reviewed by Thompson (1964) imply the more general assumption that the shear-stress *profile* (or the shear-stress integral, or the entrainment rate) depends only on the mean velocity profile, the Reynolds number and the pressure gradient. It was argued by Bradshaw & Ferriss (1965*b*) that the shear stress $\tau \equiv -\rho\overline{uv}$ is closely related to the turbulent kinetic energy $\frac{1}{2}\rho(\overline{u^2}+\overline{v^2}+\overline{w^2})$ and that the latter, being governed by the turbulent energy equation,‡ is certainly not determined uniquely by the local mean-flow conditions. This argument is not of course new: one of the first to propose it was Dryden (1946). The poor performance of the above-mentioned calculation methods in practice lends support to the view that the assumption of a close relation between the shear-stress profile and the mean velocity profile is not a realistic one for boundary layers in arbitrary pressure gradients, because it ignores the effect of the past history of the boundary layer, and that it is therefore unsuitable as the basis of the empirical correlations which must inevitably be used in any method of calculating turbulent flows short of a numerical solution of the Navier–Stokes equations.

In the present paper we show that there is a much closer connexion between

† 'Sandwich' course student, March–September 1965.

‡ Not to be confused with the mean kinetic energy equation for $\frac{1}{2}\rho(U^2+V^2+W^2)$.

the shear-stress profile and the other parameters describing the turbulence structure than between the shear-stress profile and the mean velocity profile. In fact, good results are obtained by assuming that the turbulent intensity is directly proportional to the local shear stress, that the dissipation rate is determined by the local shear stress and a length scale depending on y/δ, and that the energy diffusion is directly proportional to the local shear stress with a factor depending on the maximum value of shear stress, but the method to be described is *not* dependent on these simple choices or indeed on the assumption of a one-to-one correspondence between the shear-stress profile and the turbulence structure: this latter is simply a first approximation, which is believed to be a very much better first approximation than the assumptions made in the older methods.

If the relations between the shear stress and the intensity, dissipation and diffusion are known, the turbulent energy equation (which is an equation for the rate of change of turbulent intensity along a streamline) can be converted immediately into an equation for the rate of change of turbulent shear stress along a mean streamline in a two-dimensional boundary layer (the possibility of an extension to three-dimensional flow is discussed in § 6). This equation, together with the mean momentum equation (the 'boundary-layer equation') and the mean continuity equation, can be solved numerically by a step-by-step method. Even with the present meagre information about the empirical functions relating the shear stress to the other properties of the turbulence, the results compare favourably with the predictions of the integral methods and the new method seems inherently less likely to give ridiculous results in the special cases which bring the integral methods to grief. An electronic digital computer is needed but the cost of KDF 9 computer time, using a rather slow-running programming language, is only about 2*s*. 6*d*. for a streamwise distance of one boundary-layer thickness and could be reduced to about 2*d*. by writing the program in machine code. The cost on other machines should be of the same order, so that the length of the calculations compared with the integral methods is not prohibitive.

The method is a logical extension of the work of Townsend (1961, 1965), who used the turbulent energy equation to produce an equation for the shear stress in the inner layer where the rate of change along a mean streamline is negligible. In fact Townsend, in the appendix to the latter paper, anticipated the present method by making an allowance for the advection. The method also bears some resemblance to the work of Glushko (1965), although Glushko uses the turbulent kinetic energy equation only to calculate the turbulent intensity $\overline{q^2}$ and then makes the highly questionable assumption that the eddy viscosity ν_T is given by $(\overline{q^2})^{\frac{1}{2}} L/\nu_T = \text{constant}$. Prof. G. M. Lilley has informed us that he has done some work (unpublished) on similar lines to ours.

In § 2, the shear-stress equation is derived: it is shown that the shear-stress equation, the momentum equation and the continuity equation form a hyperbolic system, and the physical significance of this is discussed. In § 3, estimates are made of the three empirical functions which appear in the shear-stress equation. § 4 presents the numerical method using the method of characteristics, and in § 5 some comparisons with experiment are shown. The computer program is described by Ferriss & Bradshaw (1966), and a much faster one is being developed.

2. Derivation of a differential equation for turbulent shear stress

The turbulent energy equation for a two-dimensional incompressible mean flow, outside the viscous sublayer, is (Townsend 1956)

$$\tfrac{1}{2}\rho\left(U\frac{\partial\overline{q^2}}{\partial x}+V\frac{\partial\overline{q^2}}{\partial y}\right)-\tau\frac{\partial U}{\partial y}+\frac{\partial}{\partial y}(\overline{pv}+\tfrac{1}{2}\rho\overline{q^2v})+\rho\epsilon=0, \tag{1}$$

advection — pro-duction — diffusion — dissipa-tion

where $q^2=u^2+v^2+w^2$, $\tau=-\rho\overline{uv}$ and $\epsilon\simeq\nu\overline{(\partial u_i/\partial x_j)^2}$.

It can be regarded as an equation for the advection or rate of change of turbulent kinetic energy along a mean streamline through a point if all the other terms are known at that point, just as the boundary-layer momentum equation,

$$\rho[U\partial U/\partial x+V\partial U/\partial y]=\rho U_1\,dU_1/dx+\partial\tau/\partial y, \tag{2}$$

can be regarded as an equation for the rate of change of mean-flow momentum ρU.

Now, if we define

$$\left.\begin{aligned} a_1&\equiv\tau/\rho\overline{q^2},\quad L\equiv(\tau/\rho)^{\frac{3}{2}}/\epsilon,\\ G&\equiv\left(\frac{\overline{pv}}{\rho}+\tfrac{1}{2}\overline{q^2v}\right)\Big/\left(\frac{\tau_{\max}}{\rho}\right)^{\frac{1}{2}}\frac{\tau}{\rho},\end{aligned}\right\} \tag{3}$$

(the definition of G being chosen for later convenience), (1) becomes

$$U\frac{\partial}{\partial x}\left(\frac{\tau}{2a_1\rho}\right)+V\frac{\partial}{\partial y}\left(\frac{\tau}{2a_1\rho}\right)-\frac{\tau}{\rho}\frac{\partial U}{\partial y}+\left(\frac{\tau_{\max}}{\rho}\right)^{\frac{1}{2}}\frac{\partial}{\partial y}\left(G\frac{\tau}{\rho}\right)+\frac{(\tau/\rho)^{\frac{3}{2}}}{L}=0, \tag{4}$$

where a_1, L and G are functions of y/δ which depend on the shape of the shear-stress profile. a_1 and G are dimensionless and L has the dimensions of length: it is nearly the same as the 'dissipation length parameter' L_ϵ used by Townsend (1961), and is the most important of the three functions because over most of the boundary layer the dissipation is much larger than the advection or diffusion. By dividing through by τ/ρ it can be seen that (4) is the 'mixing-length' equation with additional terms representing advection and diffusion. The present method is therefore a refinement of mixing-length theory, and reduces to mixing-length theory in regions where the latter is valid according to the analyses of Batchelor (1950) and Townsend (1961). If adequate assumptions can be made for a_1, L and G then (2) and (4), together with the continuity equation

$$\partial U/\partial x+\partial V/\partial y=0 \tag{5}$$

form a set of three equations in the three unknowns U, V and τ (the computer program covers the case of laterally convergent boundary layers and thin boundary layers on bodies of revolution but for simplicity the consequent modi fications to the analysis are omitted from the present paper).

With the exception of the pressure diffusion $\overline{pv}$, which, as Bradshaw & Ferriss (1965*a*) and Bradshaw (1966) have suggested on slender evidence, may be small,

all the terms in the definitions of a_1, L and G can be measured by hot-wire techniques. Although such measurements are not at present of high accuracy they can be used as a first approximation, to be improved by comparing the results of numerical calculations of boundary-layer development with experimental results. Although it is the chief merit of the present method that it takes into account the upstream history of the flow as represented by the advection term, this term is sufficiently small in most cases for the exact behaviour of a_1 to be less important than the behaviour of L, except in the outer part of the boundary layer where advection and diffusion are nearly equal, and even here it is only necessary to insert the correct behaviour of $a_1 G$ (see (17) below). For most of these early calculations we have taken a_1 to be an absolute constant, 0·15, and the analysis below will be presented for constant a_1: the hyperbolic nature of the equations is not altered by allowing a_1 to vary and the numerical procedure is insignificantly more complicated, but the details are less easy to follow. The choice of the empirical functions is discussed in § 3.

The hyperbolicity of (2), (4) and (5) can be demonstrated by standard methods (see § 4). Mathematically, hyperbolicity means that there exist real 'characteristic' directions, as many in number as there are equations, along which the partial differential equations reduce to ordinary differential equations containing gradients *along* the characteristics only. In the present case, the angle between a characteristic and the x-axis, γ, is given by the three values

$$\tan\gamma = \infty,\ [V + a_1 G \tau_{\max}^{\frac{1}{2}} \pm (a_1^2 G^2 \tau_{\max} + 2a_1\tau)^{\frac{1}{2}}]/U, \tag{6}$$

(in this and the following analysis we write τ for τ/ρ, for simplicity).

In any system of equations based on the boundary-layer approximation, at least one characteristic will be vertical because pressure disturbances are assumed to propagate instantaneously through the thickness of the boundary layer ($\partial p/\partial y = 0$), even in supersonic flow.

It is easily seen by substituting the continuity equation into the momentum equation that the equation along the vertical characteristic is

$$-U/dV/dy + V/dU/dy = U_1/dU_1/dx + d\tau/dy. \tag{7}$$

The equations along the inclined characteristics are

$$\tau\frac{dU}{ds} - \frac{1}{2}\left(GM \pm \left(G^2M^2 + \frac{2\tau}{a_1}\right)^{\frac{1}{2}}\right)\frac{d\tau}{ds} = \frac{\tau}{U}\left[U_1\frac{dU_1}{dx} + a_1\left(\frac{\sqrt{\tau}}{L} + M\frac{dG}{dy}\right)\left(GM \pm \left(G^2M^2 + \frac{2\tau}{a_1}\right)^{\frac{1}{2}}\right)\right]\frac{dx}{ds}, \tag{8}$$

where s is measured along the characteristic, dx/ds is unity to the boundary-layer approximation, and $M \equiv \tau_{\max}^{\frac{1}{2}}$.

When (2), (4) and (5) were first derived, we supposed, in so far as we thought about it at all, that the equations were parabolic like the ordinary laminar boundary-layer equations in which τ is given by $\tau = \mu\,\partial u/\partial y$. We are greatly indebted to Mr P. G. Williams, formerly of Mathematics Division, NPL, for pointing out that the equations were in fact hyperbolic. If a 'gradient diffusion' form is used to represent the energy diffusion a term $\partial^2\tau/\partial y^2$ appears and the equations become

parabolic: with no diffusion at all, they are hyperbolic. In § 3 it is suggested that the present form of the diffusion term is physically more reasonable than a gradient-diffusion form.

It is seen that the equations along the inclined characteristics do not contain V at all, so that, to the boundary-layer approximation, the effect of the V-component velocity is merely to displace the flow in the y-direction, leaving the development of U and τ along a given streamline unaltered. A related but less general principle holds in laminar boundary layers, where it is called Prandtl's transposition theorem. V occurs in the equations for the characteristic directions: when G is small, in the inner layer,

$$\tan\gamma = [V \pm (2a_1\tau)^{\frac{1}{2}}]/U \tag{9}$$

and the characteristics are at equal and opposite angles to the mean streamline. When τ is small, near the outer edge,

$$\tan\gamma = V/U,\ (V + 2a_1 G\tau_{\max}^{\frac{1}{2}})/U \tag{10}$$

so that one characteristic coincides with the streamline and the other is inclined at a steeper angle than the streamline because of the outward diffusion of turbulent energy. In fact, it can be seen from (17), below, that the latter characteristic coincides with the edge of the boundary layer (defined as the line along which $U/U_1 = \text{constant} = 0{\cdot}995$, say).

The physical significance of hyperbolicity is that the effect of a small disturbance at a point P is confined to the 'downstream' side of the characteristics through P. In the familiar case of inviscid supersonic flow there are only two characteristic directions (the dependent variables being U and V), which are inclined at the Mach angles. In the present case, the variables U and τ can be considered separately from V, and the physical situation represented by the finite angle between the inclined characteristics is the finite angle of spread of contaminant diffusing from a point source. In homogeneous turbulence the standard deviation of such a contaminant 'wake' is $[(\overline{v^2})^{\frac{1}{2}}/U]\,(x-x_0)$ for small $(x-x_0)$. We should not expect any quantitative connexion between passive-scalar diffusion and the diffusion of momentum which the present equations represent: the comparison is offered as a simple explanation of the unexpected hyperbolicity of the equations. (It may be noted that in molecular diffusion the 'wake' initially spreads parabolically instead of linearly, which is why the laminar boundary-layer equations are—in the other sense—parabolic.) The only previous recognition of the hyperbolicity of turbulent diffusion that we have been able to find is in the work of Davies (1952), which does not seem to have been followed up. Monin's (1959) assumption of a finite maximum velocity of diffusion naturally leads to a hyperbolic equation but it is not necessary to make such an assumption to explain the finite rate of increase of standard deviation of the near-Gaussian contaminant profile, as has been recognized by Pasquill (1966). It should be emphasized that hyperbolicity is a feature only of the *mean* equations (2), (4) and (5): the instantaneous (Navier–Stokes) equations are elliptic.

The only vital approximations made in the derivation of the equations are (i) the boundary-layer approximation, which implies that there is no static-pressure difference across the layer, and (ii), implicitly, that a_1, L and G change much more

slowly than τ and U in the x-direction, so that they can be regarded as well-behaved *coefficients* rather than variables. The first assumption precludes upstream influence of pressure changes but the insensitivity of the development along the streamlines to changes in streamline angle leads one to believe that the hyperbolic nature of turbulent mixing is rather more fundamental than the boundary-layer approximation and is not necessarily incompatible with pressure changes across the layer or even fully elliptic behaviour of the pressure. The second assumption probably requires only that sudden changes at one value of y (for instance, a change in boundary condition at the surface) shall not produce large changes in a_1, L and G at other values of y. This seems to be another restriction on the pressure field, this time a requirement that extraneous pressure fluctuations shall not have a large effect on the shear-stress-producing part of the turbulence: some evidence in support of this was given by Bradshaw (1965). Strictly speaking, the dependence of G on $\tau_{\max}$ implies that conditions at one value of y affect conditions at all other values of y, whereas the physical reasoning advanced in support of the hyperbolicity of the model equations implies that changes should spread slowly across the boundary layer as the eddies move downstream. However, $\tau_{\max}$ is the simplest measure of the overall turbulent intensity, and will not, as a rule, change very violently with distance downstream.

Since a_1, L and G must be specified numerically, there is little hope of deriving any general analytical solutions of the equations, but two special cases are of interest.

(i) Near the surface, the turbulent energy equation reduces to 'production' equals 'dissipation' or, in the present notation,

$$\tau \partial U/\partial y = \tau^{\frac{3}{2}}/L, \tag{11}$$

so that

$$\partial U/\partial y = \tau^{\frac{1}{2}}/L \tag{12}$$

and L is equal to the mixing length, which is known to be about $0{\cdot}4y$ in this region. This special case, and its extension to the case where diffusion is not negligible, have been discussed by Batchelor (1950) and Townsend (1961, 1965).

(ii) Near the outer edge of the boundary layer, the turbulent energy equation reduces to 'diffusion' equals 'advection' so that

$$\left(U\frac{\partial}{\partial x}+V\frac{\partial}{\partial y}\right)\frac{\tau}{2a_1} = -\frac{\partial}{\partial y}(G\tau\tau_{\max}^{\frac{1}{2}}) \tag{13}$$

and, if we introduce a variable η such that

$$U\frac{\partial}{\partial x}+V\frac{\partial}{\partial y} = (V-\eta U)\frac{\partial}{\partial y}, \tag{14}$$

which is in effect an assumption of self-preservation, we have

$$(V-\eta U)\frac{\partial}{\partial \eta}\frac{\tau}{2a_1} = -\frac{\partial}{\partial \eta}(G\tau\tau_{\max}^{\frac{1}{2}}). \tag{15}$$

Over a small range of η in the region where τ goes to zero very rapidly, we can integrate to obtain

$$\eta U - V = 2a_1 G\tau_{\max}^{\frac{1}{2}}, \tag{16}$$

and since, at $y = \delta$, $\eta U - V$ is equal to the entrainment rate $dQ/dx \equiv d U_1(\delta - \delta_1)/dx$ we have

$$dQ/dx = 2\tau_{\max}^{\frac{1}{2}}(a_1 G)_{y=\delta}. \tag{17}$$

These two special cases are useful in choosing the empirical functions for L and $a_1 G$: it is far easier to measure entrainment from mean velocity profiles than to measure the turbulent diffusion directly. The identification of L with the mixing length in the region where advection is negligible implies that all the results of mixing-length theory for the inner layer (see, for instance, Townsend (1965)) can

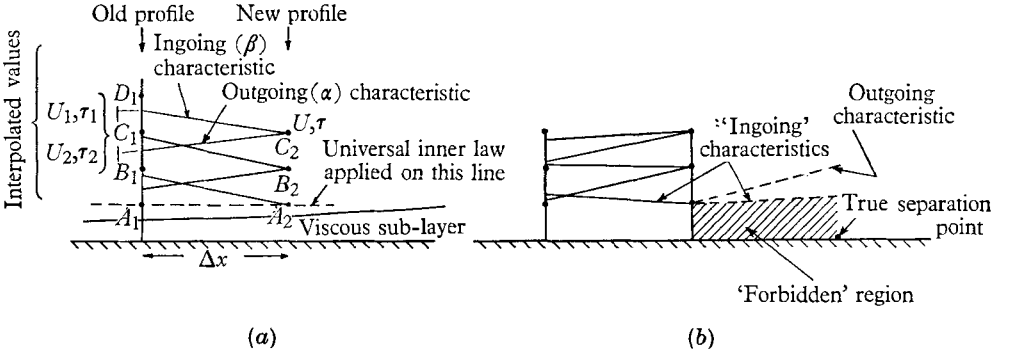


FIGURE 1. (*a*) Characteristic mesh. (*b*) Near separation.

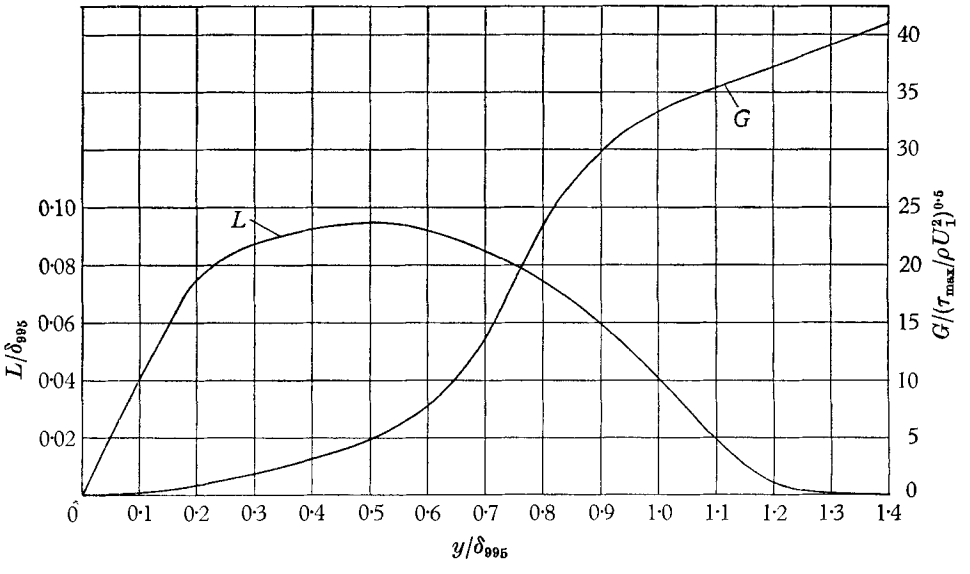
be recovered, sometimes more elegantly, and that the numerical solutions can be joined on to the universal inner-layer solutions for a smooth, rough or porous surface. It should be emphasized that L is *not* equal to the mixing length if advection or diffusion is appreciable: the distinction between the present theory and mixing-length theory is that L, which is a quantity relating one turbulence parameter to another, is much more likely to be a universal function than the mixing length, which is a quantity relating a turbulence parameter to the mean flow. Therefore the present calculation method can scarcely fail to be an improvement over the mixing-length (or eddy-viscosity) methods.

The behaviour of the equations near separation is interesting. If $d\tau_w/dx$ is negative and large enough, the value of V at the calculation point nearest the surface will exceed $\sqrt{(2a_1\tau)}$ so that the 'ingoing' characteristic moves outwards (figure 1 (*b*)). Once this has happened, the region between this characteristic line and the wall is inaccessible. In practical cases this occurs extremely near the true separation point and is of no consequence; in fact V is taken as zero at the first calculation point for computational simplicity. The same phenomenon occurs in boundary layers with injection.

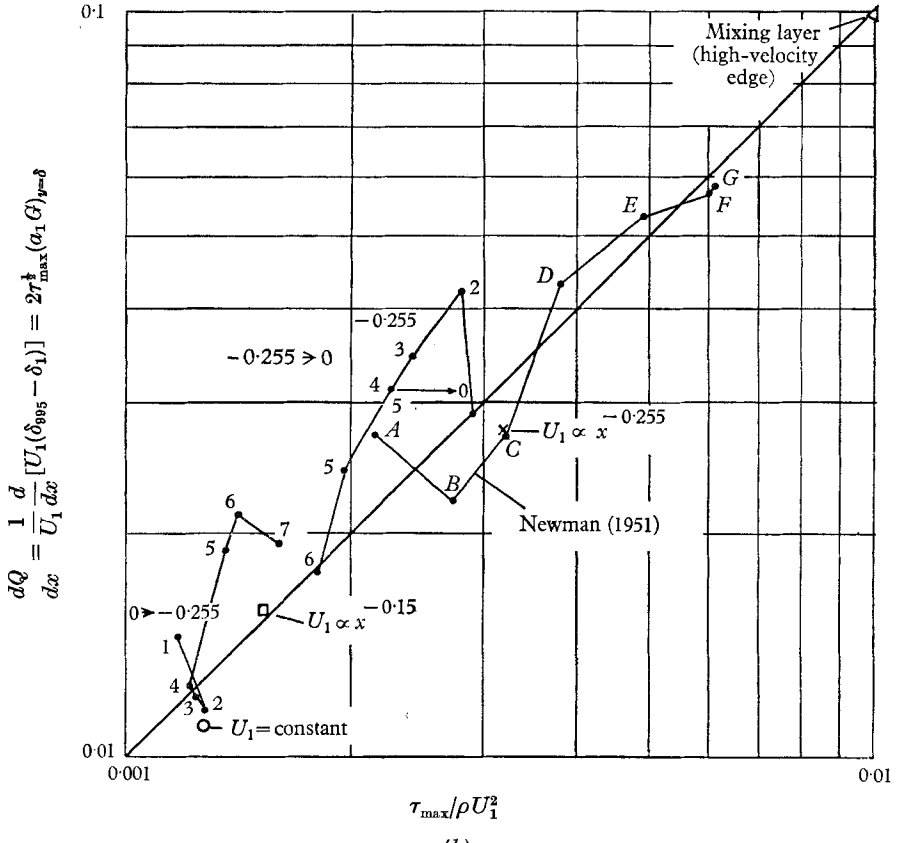
3. The choice of empirical functions

The accuracy of the calculation method depends entirely on the empirical functions a_1, L and G. The moderately good results reported in the present paper have been achieved with extremely simple choices

$$a_1 = 0{\cdot}15, \quad L/\delta = f_1(y/\delta), \quad G = (\tau_{\max}/U_1^2)^{0.5} f_2(y/\delta),$$



(*a*)



(*b*)

FIGURE 2(*a*), (*b*). Empirical functions used in the calculations.

where δ is the total thickness of the boundary layer: strictly, the length scale should be a typical length scale of the largest turbulent eddies, for which the most logical choice is the distance from the surface to the point at which the intermittency, γ, is 0·5 (the average position of the turbulence front), but the ratio of this distance to the boundary-layer thickness seems to be nearly constant, at least for equilibrium boundary layers (Bradshaw 1966). We take δ as the value of y at which $U/U_1 = 0{\cdot}995$, the definition used by Head (1958) and others.

The dissipation length parameter L varies as Ky near the wall, where K is von Kármán's constant, and is satisfactorily represented in the outer part of the boundary layer by $0{\cdot}095\delta\gamma^{\frac{1}{2}}$, which implies that the dissipation length parameter within the turbulent fluid, $(\gamma\tau)^{\frac{3}{2}}/\gamma\epsilon$, is independent of y in the outer part of the layer. Direct evidence for this is given by Bradshaw (1966): in that paper it is shown that the higher-frequency part of the spectra in the outer part of three widely different equilibrium boundary layers is determined, to fair accuracy, by a velocity scale $\tau^{\frac{1}{2}}$ and a length scale δ. If, as is to be expected, the energy transfer through this frequency range to the very-high-frequency dissipating eddies is determined by the same scales, it follows that $\gamma\epsilon \propto (\gamma\tau)^{\frac{3}{2}}/\delta$. An analogous argument in the inner layer, based on the spectra of Bradshaw (1965), which scale on $\tau^{\frac{1}{2}}$ and y, leads to the well-supported results $L \propto y$, which lends support to the above deductions for the outer layer.

In the last-named paper, evidence is presented that the considerable variation in a_1 observed experimentally is at least partly due to an 'inactive', quasi-irrotational component of the turbulent motion (Townsend 1961), which does not contribute to the shear stress or the dissipation and can therefore be disregarded: therefore $a_1 =$ constant is a much better approximation than at first appears. The value of a_1 in an infinite uniform shear flow (Rose 1966) is about 0·17.

The definition of the diffusion function G requires some justification, since turbulent diffusion is still often thought of as a gradient-diffusion process and Glushko, for instance, has assumed that

$$\overline{pv} + \tfrac{1}{2}\rho\overline{q^2 v} \propto y(\partial/\partial y)\,(\overline{q^2})^{\frac{3}{2}}.$$

Townsend (1956) has argued that at least part of the turbulent diffusion of energy is effected by the large eddies, and more recent studies (see, for instance, Bradshaw (1966)) have indicated that the large eddies are much stronger than Townsend supposed, so that it seems probable that most of the diffusion is effected by the large eddies, at least in the outer part of the boundary layer where the diffusion term in the energy equation is most important. This part of the diffusion may be represented by

$$\overline{pv} + \tfrac{1}{2}\rho\overline{q^2 v} = \tfrac{1}{2}\rho\overline{q^2\mathscr{V}},$$

where $\mathscr{V}$ is the effective velocity at which the turbulent energy is transported in the y-direction by the large eddies. Now $\overline{\mathscr{V}^2}$ is expected to be related to the average shear stress across the outer part of the boundary layer, since this is what will determine the large-eddy intensity, and an adequate preliminary choice is the maximum shear stress in the region $0{\cdot}25\delta_{995} < y < \delta_{995}$; we expect, of course, that $\mathscr{V}$ will be a function of y/δ. Thus, replacing the turbulent energy by τ, we get

$$\overline{pv} + \tfrac{1}{2}\rho\overline{q^2 v} = \tau(\tau_{\max})^{\frac{1}{2}}\,G(y/\delta).$$

Keffer (1965) has represented the energy diffusion in a wake by the sum of a large-eddy contribution and a gradient-diffusion contribution, which he suggests will be of equal order. It can be seen that the gradient-diffusion term leads to an improbable variation of velocity near the edge of the flow, but further experience with the present calculation method may make it advisable to insert a gradient-diffusion term to represent processes well inside the turbulent flow.

The present hypothesis indicates that the dimensionless entrainment rate, related to the energy diffusion by (17), should vary as $\tau_{\max}^{\frac{1}{2}}/U_1$ if the ratio of the large-eddy intensity to $\tau_{\max}$ is constant. In fact Bradshaw (1966) has shown that in equilibrium boundary layers the large-eddy intensity increases more quickly than $\tau_{\max}$: the more intense the turbulence, the stronger the large eddies are in comparison with the rest of the turbulence. The ultimate is the free mixing layer (Bradshaw, Ferriss & Johnson 1964) where the large eddies entirely dominate the turbulence. In figure 3(*b*) values of entrainment rate are shown for the mixing layer and for several very different boundary layers. The boundary layers marked '$0 \rightarrow -0{\cdot}255$' (Bradshaw, to be published) and 'Newman (1951)' are progressing towards separation while '$-0{\cdot}255 \rightarrow 0$' (Bradshaw & Ferriss 1965*b*) passes from adverse pressure gradient to zero pressure gradient. In view of the difficulties of measuring entrainment (we have allowed for lateral convergence in some of these flows) the most that can be said is that the results give fair support for the relation

$$\frac{1}{U_1}\frac{dQ}{dx} = 10\left(\frac{\tau_{\max}}{U_1^2}\right)^{1.0},$$

and it follows from (17) that $G(y/\delta)$ itself varies as $(\tau_{\max}/U_1^2)^{0.5}$. Where the shear stress is a maximum at or close to the wall we have taken $\tau_{\max}$ as τ at $y/\delta = 0{\cdot}25$, a value chosen to give the right entrainment in Herring & Norbury's (1963) accelerating boundary layer. It may be remarked that this empirical variation of G with $\tau_{\max}/U_1^2$ is the only information needed in the calculations that is not derived solely from the boundary layer in zero pressure gradient.

The functions used in the calculations presented here are shown in figure 3(*a*). a_1 was taken as 0·15. The calculations are extraordinarily insensitive to the behaviour of G within the layer and even a change in G at the outer edge of the layer produces a much less than proportional change in the results. The sensitivity to changes in L is roughly what would be expected from assuming L to be equal to the mixing length.

4. The numerical method

Before discovering that the equations were hyperbolic we programmed an explicit rectangular-mesh finite-difference method. Although it was abandoned in favour of the method of characteristics it may prove to have some uses in the future: a brief description was given in the unpublished version of this paper, NPL Aero Report 1182 (1966).

The derivation of the equations for the three characteristic angles and the differential equations along the characteristics is tedious but straightforward. Good accounts of the method used are given by Ralston & Wilf 1960, chapter 15),

Collatz (1959, p. 323) and Staff of the Mathematics Division NPL (1961, chapter 11). The first-named is especially recommended from the point of view of numerical solution by a digital computer. The characteristic angles are given by (6) and the differential equations are (7) and (8). Since one of the characteristics is the line x = constant, on which it is convenient to specify the initial values, only two of the three variables need be specified, the third being derivable from the equation along this characteristic. We have chosen to specify U and τ.

V does not appear in (8), so that, after conversion to finite-difference form, only two simultaneous equations have to be solved, for U and τ, and not three for U, V and τ. (7) is then integrated separately to provide V, which appears in the equations for the slopes of the inclined characteristics, $\tan\alpha$ and $\tan\beta$ say. The finite-difference scheme used calculates new values at equal intervals along a line x = constant rather than at points entirely determined by the characteristic mesh ('Method II' of Ralston & Wilf).

Figure 1 (*a*) shows how U, V and τ on a new profile are calculated from the values on the previous profile. The line labelled 'old profile' represents either the input or the last profile calculated at any stage during the run. The calculation is started at a finite distance from the wall because the equations are not valid in the viscous sublayer—indeed the equations have a singularity at the wall, where all three characteristic angles tend to 90°. It is assumed that U and τ at the first mesh point (A) are connected by the modified logarithmic law

$$U = \frac{u_\tau}{K}\left(\log_e \frac{u_\tau y}{\nu} + \mathrm{A}\right) + \frac{u_\tau}{K}\left[2\log_e\left(\frac{2}{(1+z)^{\frac{1}{2}}+1}\right) + 2((1+z)^{\frac{1}{2}}-1)\right], \qquad (18)$$

with $K = 0{\cdot}40$, $A = 2{\cdot}0$, where $z = \alpha y/\tau_w$ and the relation $\tau = \tau_w + \alpha y$ is assumed to hold near the wall, and where α is determined from the previous x-steps. This enables a solution at point A_2 downstream (see figure 1 (*a*)) to be obtained from the equation along the β or 'ingoing' characteristic only. The value of τ_w actually output by the computer is derived from a new value of α obtained from a least-squares straight line through the first three points on the τ profile.

We have abandoned our earlier requirement that $u_\tau y/\nu$ at point A should be greater than 40, so that the logarithmic law should be valid at that point, because it leads to inaccuracy and instability in extrapolating the shear stress to the surface at low Reynolds numbers when point A has to be taken several y-steps from the surface. Taking this distance to be equal to the y-step in all cases implies the assumption that the shear stress at $u_\tau y/\nu < 40$ is the same as if the logarithmic law was still valid—which is at least as justifiable as assuming $\tau = \tau_w + \alpha y$. The velocity at point A output by the computer is incorrect if $u_\tau(y \text{ step})/\nu < 40$ but the displacement and momentum thicknesses are evaluated correctly by our program.

In the case of boundary layers with suction or injection the 'logarithmic' law used as the inner boundary conditions becomes

$$\frac{2}{V_w}[(UV_w + u_\tau^2)^{\frac{1}{2}} - u_\tau] = \frac{1}{K}\left[\log_e \frac{u_\tau y}{\nu} + A\right],$$

where A is in general a function of V_w/u_τ.

Equations (5) and (8) are re-written with finite-difference quotients to represent the derivatives in the characteristic direction, and the x-step Δx is chosen subject to a stability condition to be mentioned later. We then have for the α and β characteristic equations

$$\left.\begin{aligned} \tau_\alpha(U-U_{2c})-\tfrac{1}{2}\sigma_\alpha(\tau-\tau_{2c}) &= \Delta x\frac{\tau_\alpha}{U_\alpha}\left[U_1\frac{dU_1}{dx}+\frac{a_1}{\delta_{995}}\left(\frac{\sqrt{\tau_\alpha}}{L_\alpha}+G'_\alpha M\right)\sigma_\alpha\right] \\ \text{and}\quad \tau_\beta(U-U_{1c})-\tfrac{1}{2}\sigma_\beta(\tau-\tau_{1c}) &= \Delta x\frac{\tau_\beta}{U_\beta}\left[U_1\frac{dU_1}{dx}+\frac{a_1}{\delta_{995}}\left(\frac{\sqrt{\tau_\beta}}{L_\beta}+G'_\beta M\right)\sigma_\beta\right], \end{aligned}\right\} \tag{19}$$

where $\sigma_\alpha = G_\alpha M+[G_\alpha^2 M^2+(2\tau_\alpha/a_1)]^{\frac{1}{2}}$, $\sigma_\beta = G_\beta M-[G_\beta^2 M^2+(2\tau_\beta/a_1)]^{\frac{1}{2}}$ and U_{1c} and U_{2c} are the values of U at the ends of the characteristic segments. The α and β suffixes to U, τ, L and G denote average values of these quantities on the characteristic as distinct from end values. (19) can now be solved for U and τ.

The method proceeds as follows, starting at mesh point B on the new profile.

(1) The old values of α and β are used to construct straight-line portions of the characteristics intersecting on the new profile and cutting the old profile at intermediate points.

(2) U and τ are interpolated at these intermediate points and labelled u_1, τ_1, u_2 and τ_2 as shown in figure 1 (*a*). A quadratic interpolation procedure is used for this.

(3) The values of U and τ with α and β suffixes in (19) are put equal to u_1 and τ_1 or u_2 and τ_2 as appropriate as a first approximation.

(4) The resulting equations are then solved for U and τ on the new profile.

(5) The process is repeated at the other mesh points C, D, etc., so as to obtain U and τ over the whole range of y.

(6) All the U and τ values are then used to find corresponding V values along the vertical characteristic using a finite difference form of (3), starting with $V_A = 0$ at point A.

(7) Values of α and β are then calculated for the new profile.

(8) The values of U, τ, V, α and β on the new profile are then used to correct estimates of these quantities on the characteristics for the first 10–20 % of the boundary layer, where the gradients are largest. New values of the quantities are then re-calculated using these new estimates for this portion of the boundary layer. The alternative to this is to decrease the x-step which would result in increased computer time spent on the outer portion of the boundary layer, which is already calculated to sufficient accuracy.

(9) The new profile is then used as the input for calculations one step further downstream.

A flow diagram of the computer program is shown in figure 3.

It was mentioned above that the x-step choice is subject to one condition: this is that

$$r \equiv (\Delta x/\Delta y)\tan\alpha_{(\max)} \leqslant 1,$$

where Δy is the step size in the y-direction, usually taken as $0{\cdot}02\delta$ on the initial profile. We have found this to be necessary for the stability of the method as a whole and it has so far proved sufficient, except in the latest version of the application of the inner boundary condition, where the use of a least-squares linear

extrapolation of the shear stress to the wall caused instability for $r = 1$ in some cases: putting $r = 0{\cdot}9$ cured the trouble, at the cost of a 10 % increase in computer time. For the case of the linear wave equation it has been shown (Richtmyer 1957) that $r \leqslant 1$ is necessary *and* sufficient for stability, although this has not been proved for general non-linear hyperbolic systems as yet. The only other restrictions required for stability are that U/U_1 shall not exceed unity, and that

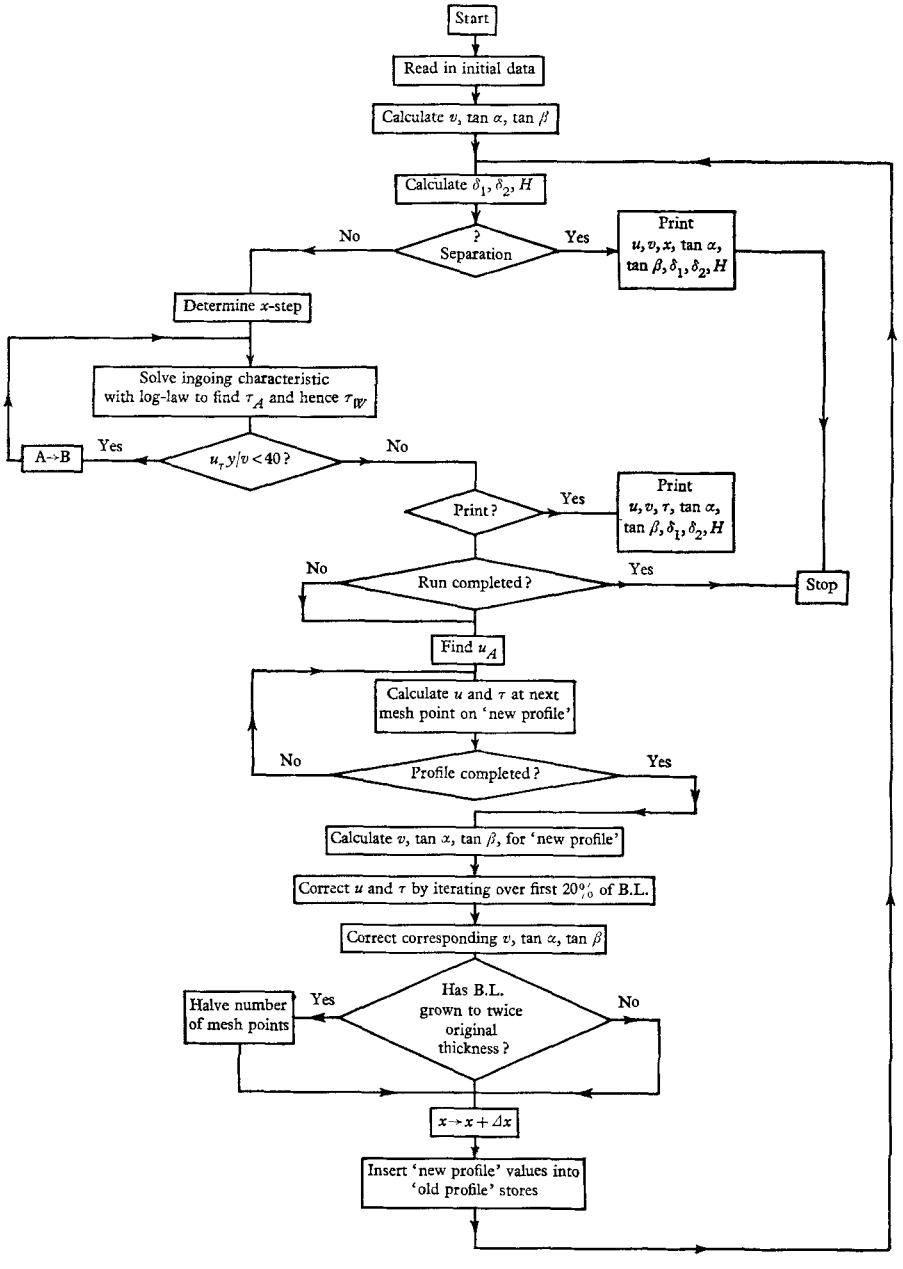


FIGURE 3. Flow diagram (see figure 1 for notation).

τ shall not be less than zero, at the edge of the boundary layer. This merely involves 'clipping' values which fall outside these limits because of finite-difference errors.

The boundary layer is allowed to grow during the calculation by adding two mesh points at every x-step, and putting U and τ at these points equal to the free-stream velocity and zero respectively. The calculation of U and τ in the y-direction is then halted whenever two consecutive U and τ values differ by less than a specified amount. This rejects any of the extra added mesh points that are not required; in fact, the boundary layer cannot grow faster than about one mesh point per x-step, because the edge of the boundary layer coincides with the outgoing characteristic. Subsequent to every x-step, the L and G data are rescaled on the new value of boundary-layer thickness. When the number of points on the profile reaches double the initial number (which is usually taken as 50), every other point is rejected and the calculation continued with twice the y-step: this saves computer time. The accuracy of the finite-difference approximation is shown by the absence of any appreciable transients resulting from this change, and by the satisfactory agreement with the momentum integral equation (less than 1 % error over a 100-step run in zero pressure gradient).

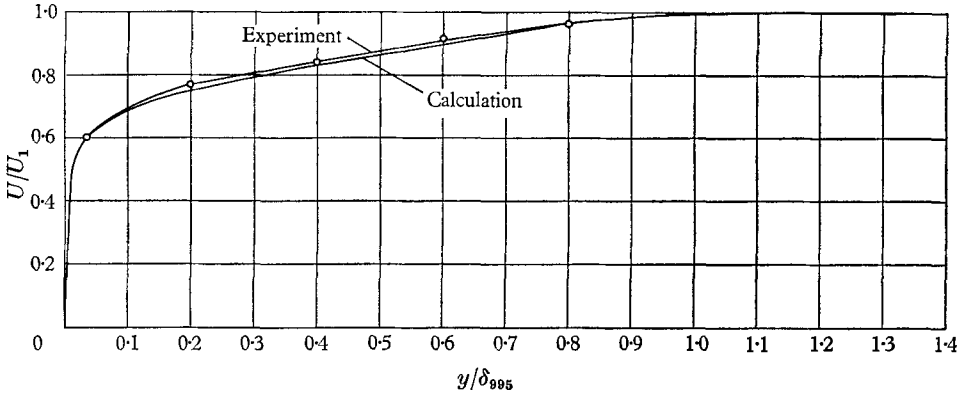
5. Test cases

We have used Klebanoff's (1955) results in zero pressure gradient to find the best choices for a_1, L and G, starting from values of these functions determined from hot-wire measurements and adjusting them by trial and error so as to get the best agreement between calculated and measured velocity and shear-stress profiles (figure 4). There is still room for slight improvement but the surface shear stress is predicted to within the current uncertainty of experimental results. The variation of surface shear stress with Reynolds number as found from runs at $U_1\delta_2/\nu = 14{,}000$ and 140,000 agrees within about 2 % with the empirical relation of Coles (1962). At the higher Reynolds number the calculated values agree with the measurements of Winter, Smith & Gaudet (1965) but at lower Reynolds numbers these experiments give a smaller shear stress than either Coles's law or the present calculations. Except for the variation of G with $\tau_{\max}/U_1^2$ the only effect of Reynolds number in the present calculation method is represented by the well-established logarithmic law, so that the method should be reliable over a wide range of Reynolds numbers without the usual dangers of extrapolating laboratory data to full-scale conditions.

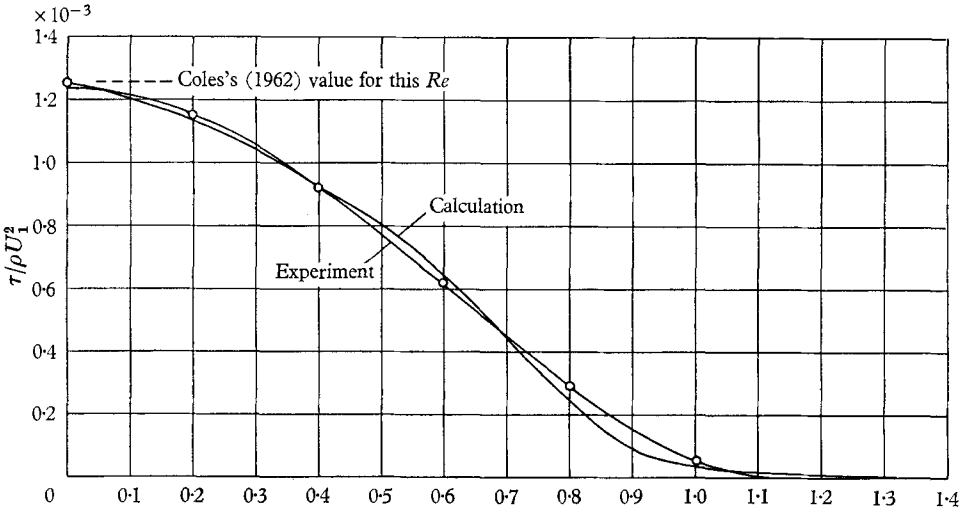
In figures 5 to 7 the calculated velocity and shear stress profiles in several equilibrium boundary layers are compared with experiment. The empirical functions used were the same as in figure 4. The agreement could be made exact by allowing the empirical functions to depend on $\tau_{\max}/U_1^2$ but the figures show that the present simple assumptions are reasonably accurate.

Figures 8(a) to 8(f) show comparisons with experiment in arbitrary pressure gradients, including two boundary layers with a prolonged region of adverse pressure gradient (Bradshaw, to be published) set up especially as a severe test of the method: the advection terms are more important in boundary layers of

this type than in those progressing rapidly towards separation. These figures show *all* the test cases we have done: we have not selected the best. Detailed tabulations are available from the authors. It may be worth reiterating that, except for the 'entrainment' correlation of figure 4 (*b*) used to scale G, which does



(*a*)



(*b*)

FIGURE 4. Zero pressure gradient, Klebanoff (1955), adjusted to $U_1\delta_2/U = 12{,}400$. (*a*) Mean velocity, (*b*) shear stress.

not play a large part in the calculations, the only empirical data used were obtained from the boundary layer in zero pressure gradient. For the most part we have restricted ourselves to boundary layers for which accurate shear stress profiles are available, because prediction of the shear stress is a more severe test than prediction of the velocity profile (it is roughly the same as prediction of the velocity *gradient*). In all cases we have used the experimental measurements, without adjustment, to start the calculations: in Schubauer & Klebanoff's (1951) boundary

layer the initial shear-stress profile was taken to be that in zero pressure gradient with the appropriate surface shear stress, but the immediate divergence of c_f and H from the experimental trend indicates that this was not quite correct.

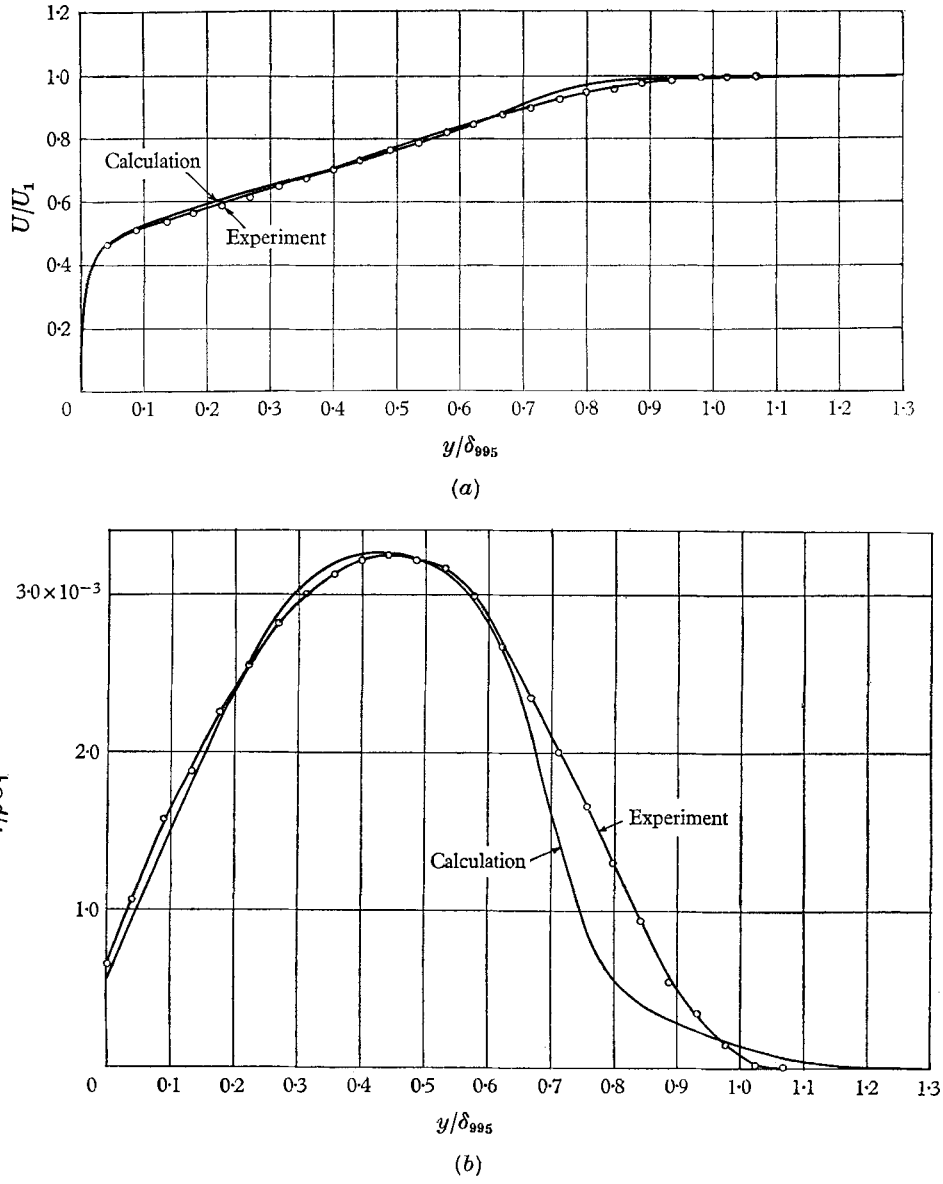


FIGURE 5. Equilibrium boundary layer $U_1 \propto x^{-0.255}$, Bradshaw (1966).
(*a*) Mean velocity, (*b*) shear stress.

Similar divergences in some of the other cases are partly attributable to shortcomings in the method and partly to inaccuracies in the starting data. For instance, the velocity profile at $x = 0$ in Schubauer & Spangenberg's (1960) boundary layer 'D' is not quite consistent with their assertion (which we have

taken to be correct) that the boundary layer developed in zero pressure gradient for $x < 0$. (The difference between the two calculations for this case shows that large errors can arise in the derivation of pressure distributions from small published graphs.) Uncertainty about the initial conditions is a very real difficulty in

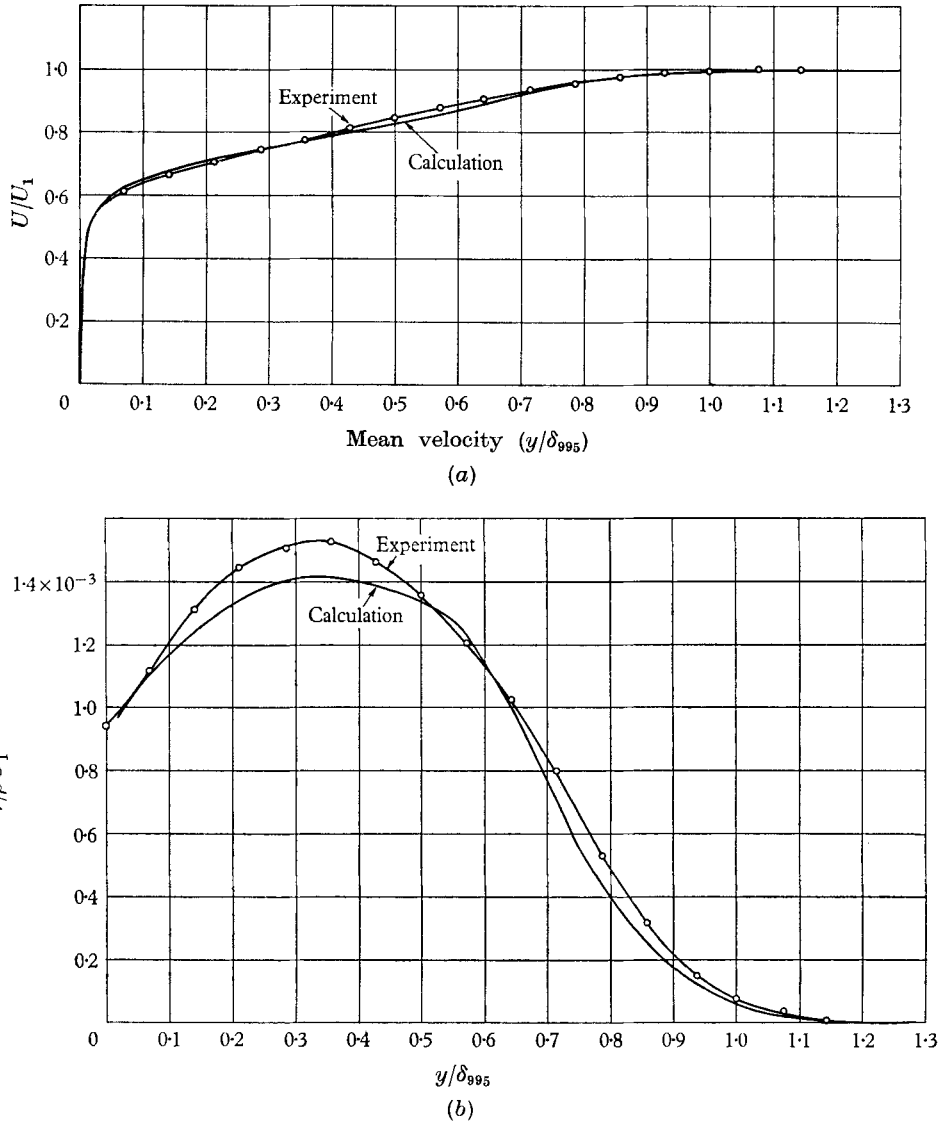
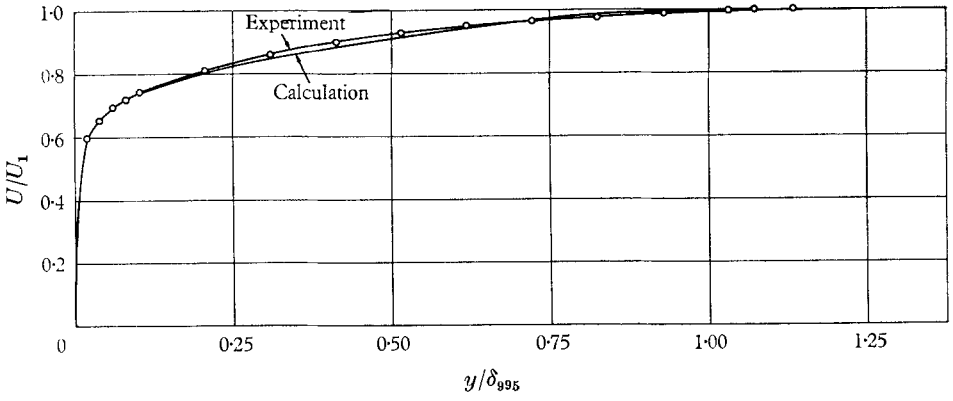


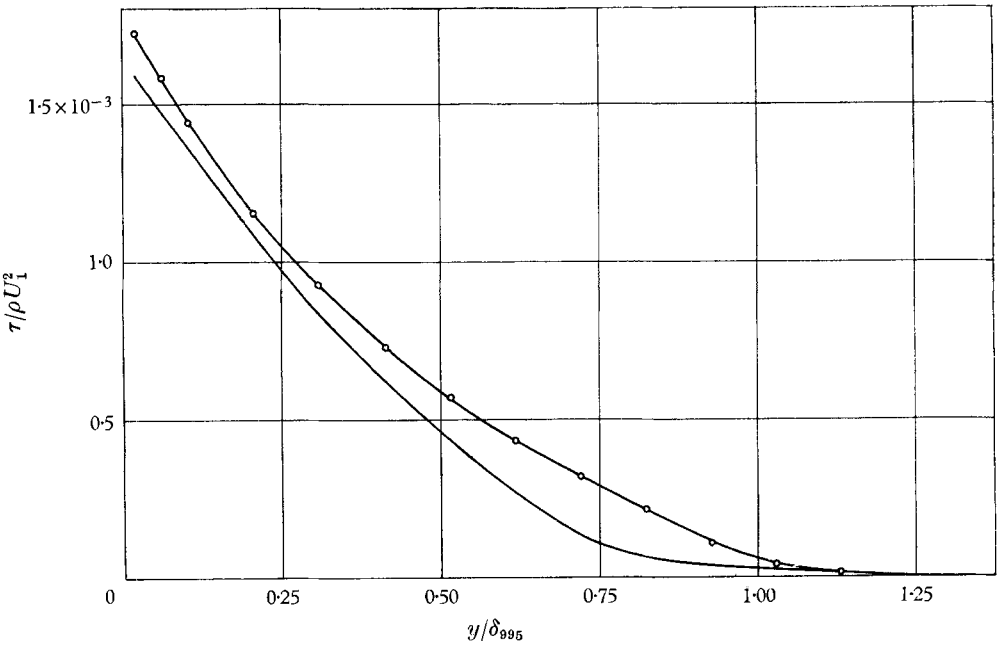
FIGURE 6. Equilibrium boundary layer $U_1 \propto x^{-0.15}$, Bradshaw (1966). (*a*) Mean velocity, (*b*) shear stress.

boundary-layer calculations (McDonald & Stoddart 1965) and we do not pretend to deal with it here. Methods which require less initial information than the present one are not to be preferred on that account: to ignore a problem is not to solve it. The velocity and shear-stress profiles are not simply related and it is necessary to insert information about both.

The only way in which we have tampered with the experimental data is to insert, in figures 8(*c*), (*d*) and (*f*), a value for the lateral convergence of the flow derived from the imbalance in the two-dimensional momentum integral equation. We have neglected the normal stress terms, whose importance cannot be



(*a*)



(*b*)

FIGURE 7. Equilibrium boundary layer in favourable pressure gradient $(\delta_1/\tau_w)dp/dx = -0{\cdot}35$, Herring & Norbury (1963). (*a*) Mean velocity, (*b*) shear stress.

estimated from velocity profiles measured with Pitot and static tubes because the effect of the turbulence on the pressure readings is of the same order as the normal stresses: it would be possible to insert an allowance for the normal-stress terms, making use of the basic assumption that the shear-stress profile uniquely deter-

mines the turbulence, but it would not be realistic to do so at present. We have also neglected variation of the pressure through the boundary layer, which may be appreciable near separation (the asterisks on the x-axis in figures 8(*c*) and 8(*d*) mark the point where V/U first exceeds 0·1 at the edge of the layer, which may be taken as the point of failure of the boundary-layer approximation). This allowance for convergence implies that the comparison between experimental

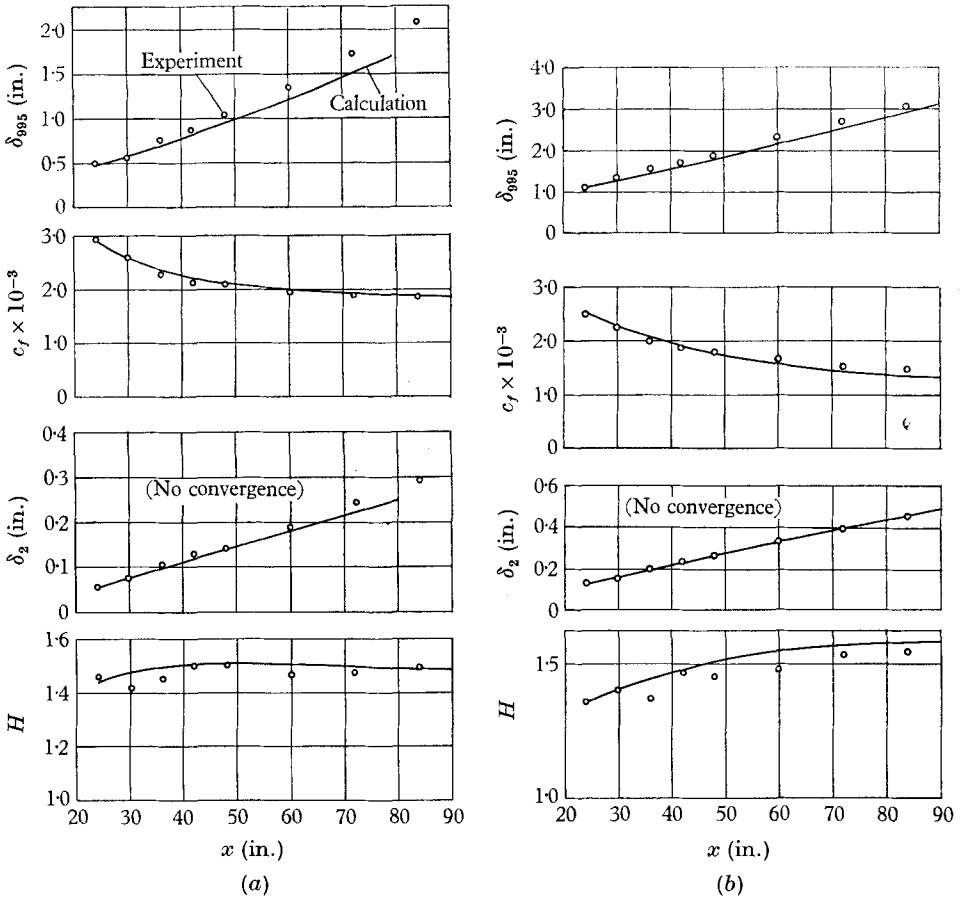


FIGURE 8. Comparisons with experiment in arbitrary pressure gradient. (*a*) Bradshaw '$0 \rightarrow -0{\cdot}255$', thin initial boundary layer, (*b*) Bradshaw '$0 \rightarrow -0{\cdot}255$', thick initial boundary layer.

and calculated values of momentum thickness δ_2 adds nothing to the comparison for c_f and H, except in some cases to show that the assumed convergence is not quite correct because of scatter in the measurements, and in others to show that an allowance for convergence *should* have been made.

H is not such an important parameter as in conventional calculation methods, since it is not used to determine c_f: for aerofoil drag prediction we merely require δ_1 and δ_2 at the trailing edge to good percentage accuracy. Nor is H, or any other parameter, required as a criterion of separation: separation is indicated by the surface shear stress becoming zero, although if separation does occur 'boundary

layer' calculations with an assumed pressure distribution are an academic exercise, and account should be taken of interaction between the boundary layer and the free stream.

The best test of the method from the engineering point of view is the prediction of c_f and δ_{995} (which is needed to scale L and G). Since there is no consistent bias in these predictions it seems likely that the method is accurate to within the

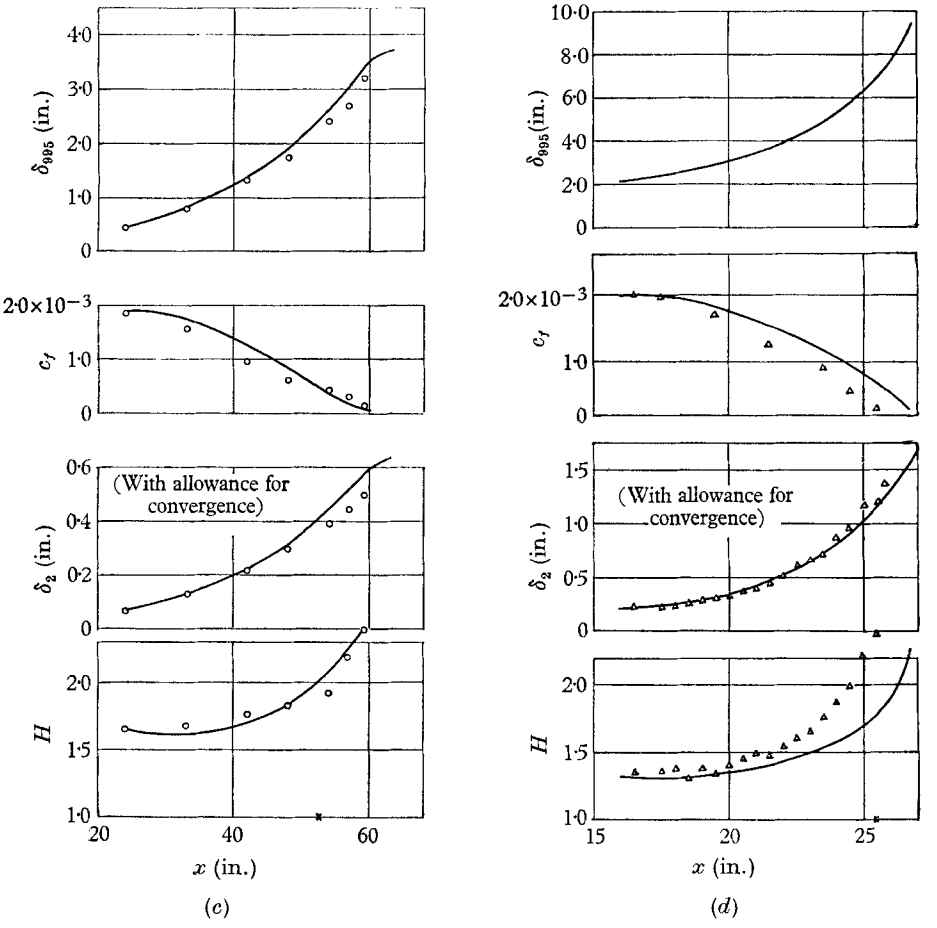


FIGURE 8. (*c*) Newman (1951), (*d*) Schubauer & Klebanoff (1951).

experimental inaccuracies (particularly the inaccuracies of the initial data and the flow convergence). This is, of course, quite a modest claim, because the data leave much to be desired. Nevertheless, the agreement with experiment over a wide range of Reynolds numbers and pressure gradients seems to be better than that of any of the existing data-correlation methods based on empirical relations between the shear-stress and velocity profiles, all of which are inevitably unreliable in boundary layers noticeably different from those used to provide the empirical data.

In addition to these test cases, we have obtained very good agreement with measurements in a boundary layer with injection ($V_w/U_1 = 0{\cdot}00178$, slight adverse

pressure gradient) by Dr J. McQuaid, formerly of the Cambridge University Engineering Laboratory, to whom we are indebted for access to his unpublished data. The additive constant A in the inner-layer velocity profile was taken as 2·0 as on a solid surface. Calculations in a boundary layer with suction ($V_w/U_1 = -0{\cdot}00246$, zero pressure gradient) required $A = 3{\cdot}3$ to give the best

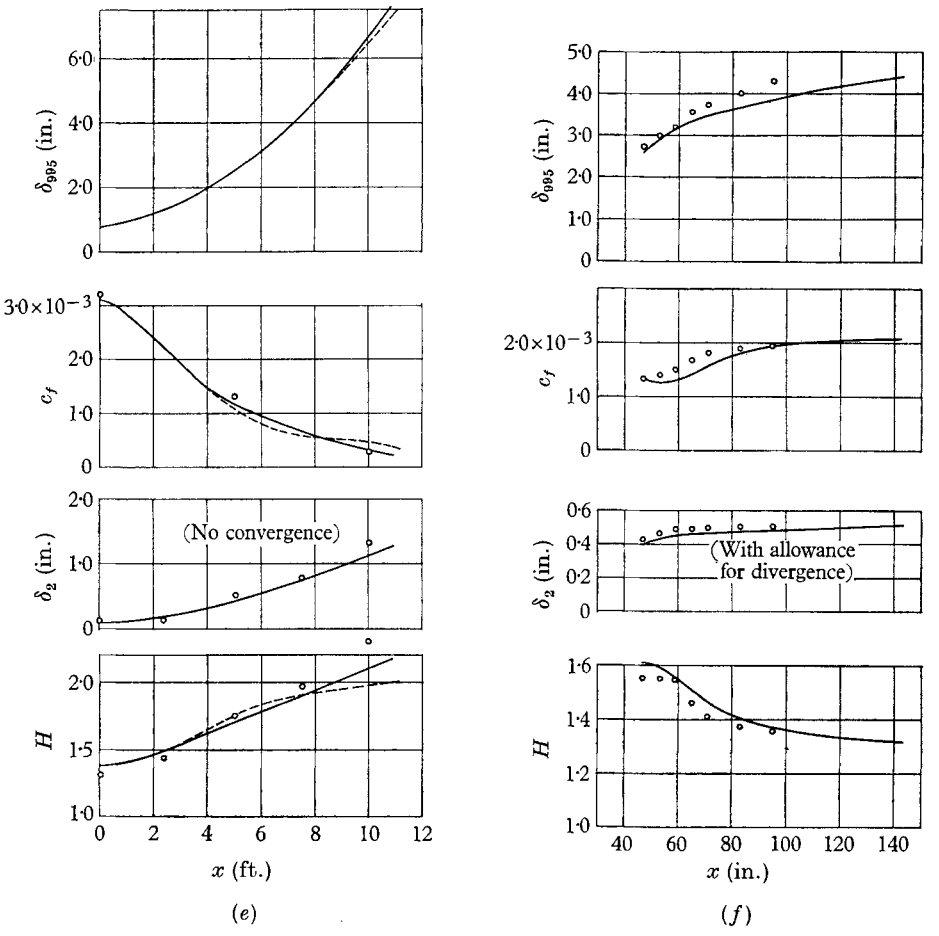


FIGURE 8. (*e*) Schubauer & Spangenberg 'D' (1960) —, calculations using pressure distribution from tabulated data; – – –, calculations using pressure distribution from published graph. (*f*) Bradshaw & Ferriss (1965*b*) '$-0{\cdot}255 \rightarrow 0$'.

agreement with the data of Sarnecki (1958, kindly supplied by Dr B. G. J. Thompson, formerly of the Cambridge University Engineering Laboratory). In view of the current uncertainty about the actual behaviour of A these calculations are not a very useful additional test of the method, although they lend support to the hypothesis that the turbulence structure is unaltered by moderate suction or injection (or, it is safe to say, by moderate surface roughness, which can be simulated in the calculations merely by prescribing A).

6. Extension of the method

The application of the method to compressible boundary layers is straightforward if one relies on Morkovin's (1962) suggestion that the turbulence structure is unaffected by compressibility if the Mach number based on a typical velocity fluctuation is small (free-stream Mach numbers less than 5 say). The extra information required is the temperature distribution across the layer and the change, if any, in the universal inner-layer velocity profile. For the case of zero heat transfer, it is adequate to use the Crocco relation for the total temperature

$$\frac{T}{T_w} = 1 - \left(\frac{T_w - T_1}{T_w}\right)\left(\frac{U}{U_1}\right)^2,$$

where suffix 1 indicates conditions at the edge of the boundary layer, or even to make the assumption of constant total temperature outside the inner layer. An empirical recovery factor is needed but this does not seem to be greatly affected by pressure gradient. Calculations by this method predict a surface shear-stress coefficient in zero pressure gradient within 3 % of the Spalding & Chi (1964) correlation up to a Mach number of 4 (Bradshaw & Ferriss 1966).

Heat transfer (with constant fluid properties) can be calculated by converting the thermal fluctuation 'energy' equation (Johnson 1959) for the mean-square temperature fluctuation $\overline{\theta^2}$ into an equation for the heat flux $\overline{\theta v}$, just as the equation for $\overline{q^2}$ is converted into an equation for the momentum flux $\overline{uv}$. Analogous empirical expressions for diffusion of temperature fluctuations by the turbulence and their dissipation by molecular conductivity are required and no data are yet available except for some measurements by Johnson which disagree rather strongly with the predictions of inner-layer similarity. The mean temperature equation and the heat-flux equation form a *pair* of hyperbolic equations with two inclined characteristics, whose failure to coincide with the inclined characteristics of the momentum equations is a measure of the departure of the turbulent Prandtl number from unity, which might be expected to be considerable in boundary layers in which 'eddy viscosity' methods give poor results.

The extension to three-dimensional boundary layers requires the specification of the *direction*, as well as the magnitude, of the shear stress. It is possible to consider the shear-stress equation in vector form and to calculate the rate of change of the shear-stress vector along a mean streamline providing that one knows the directions of the production, diffusion and dissipation vectors. It seems entirely plausible that the dissipation should act proportionally on the two components of the shear stress—that is, that the dissipation vector should always coincide with the shear-stress vector. The diffusion is sufficiently small for fairly crude empirical rules to be used, at least in mild pressure gradients. The chief difficulty is the direction of the production vector: the problem may be posed as follows. 'The rate of production of turbulent energy is the scalar product of the shear-stress vector and the principal velocity-gradient vector (the latter having components $\partial U/\partial y$ and $\partial W/\partial y$). The newly produced turbulence itself carries a shear stress. What is the direction of that shear stress ?' There is no real justifica-

tion for hoping that the question has a simple answer, but it is certainly tempting, and perhaps even reasonable, to hypothesize that the increment in shear stress is in the direction of the principal velocity-gradient vector, that is, in the same plane as the principal rates of strain. This is certainly more general than the assumption that the whole of the shear stress is in the direction of the principal velocity gradient, and reduces to that assumption in the inner layer where production equals dissipation. The contrary assumption, that the increment in shear stress is in the direction of the original shear stress, would prevent that direction from changing at all.

It is possible to derive approximate methods by using weighted integrals of the shear-stress equation across the boundary layer, together with the usual momentum and energy integral equations obtained from the mean-motion equation. Families of velocity and shear-stress profiles are required and could be obtained from 'exact' solutions of the full equations. The crudest of such methods uses an unweighted integral of the shear-stress equation (from which the diffusion disappears) and a one-parameter family of shear-stress profiles. It remains to be seen whether the large reduction in computer time will justify the inevitable decrease in accuracy.

It may be as well to end on a note of caution. The reason why the empirical relations between the shear stress and the rest of the turbulence are so well behaved is that attached turbulent boundary layers are a rather small subset of all the possible turbulent flows: the empirical expressions would not in general be valid in other types of flow such as boundary layers just downstream of reattachment, wall jets or free turbulent flows. There is no reason why different empirical relations should not be used if they covered a sufficiently large set of such flows to be useful.

We hope to explore some of these interesting and important extensions to the method ourselves, but it seems unlikely that sufficient theoretical or experimental effort will be available to do this thoroughly in the near future.

We have already mentioned our indebtedness to Mr P. G. Williams, formerly of Mathematics Division, NPL, for pointing out that our equations were hyperbolic: we have also had the benefit of several discussions with him on boundary-layer theory and numerical methods. We are indebted to several other members of Mathematics Division, in particular Mrs M. Price, for help and advice in the early stages of computer programming, and to Miss B. Webber and her staff for running the KDF 9 computer. We are grateful to a referee for several helpful comments.

The work described in this paper forms part of the research programme carried out by the Aerodynamics Division of the National Physical Laboratory for the Ministry of Aviation.

REFERENCES

BATCHELOR, G. K. 1950 *J. Aero. Sci.* **17**, 441.

BRADSHAW, P. 1965 *Nat. Phys. Lab. Aero. Rept.* no. 1172.

BRADSHAW, P. 1966 *Nat. Phys. Lab. Aero. Rept.* no. 1184, and *J. Fluid Mech.* (to be published).

BRADSHAW, P. & FERRISS, D. H. 1965*a* *Aero. Res. Counc. Lond. curr. pap.* 899.

BRADSHAW, P. & FERRISS, D. H. 1965*b* *Nat. Phys. Lab. Aero. Rept.* no. 1145.

BRADSHAW, P. & FERRISS, D. H. 1966 *Nat. Phys. Lab. Aero. Rept.* no. 1217.

BRADSHAW, P., FERRISS, D. H. & JOHNSON, R. F. 1964 *J. Fluid Mech.* **19**, 591.

COLES, D. 1962 *Rand Corp. U.S.A. Rept.* R-403-PR (see also *Aero. Res. Counc. Lond.* no. 24, 497, 1963).

COLLATZ, L. 1959 *The Numerical Treatment of Differential Equations.* Berlin: Springer.

DAVIES, R. W. 1952 *J. Appl. Phys.* **23**, 941.

DRYDEN, H. L. 1946 Paper presented at 6th Int. Congr. Appl. Mech. (see also *Nat. Adv. Comm. Aero. Wash. Tech. Note* no. 1168 (1947)).

FERRISS, D. H. & BRADSHAW, P. 1966 *Nat. Phys. Lab. Aero. Rept.* no. 1195.

GLUSHKO, G. S. 1965 *Izv. Akad. Nauk. SSSR (Mekhanika)*, July–August, p. 13 (see also *Min. of Av. T.I.L.* T/5664 (1966)).

HEAD, M. R. 1958 *Aero. Res. Counc. Lond.* R. & M., no. 3152.

HERRING, H. J. & NORBURY, J. F. 1963 Princeton University Rept. FLD-15.

JOHNSON, D. S. 1959 *J. Appl. Mech.* **26**, 235.

KEFFER, J. F. 1965 *J. Fluid Mech.* **22**, 135.

KLEBANOFF, P. S. 1955 *Nat. Adv. Comm. Aero. Wash. Rept.* no. 1247.

MCDONALD, H. & STODDART, J. A. P. 1965 *Brit. Aircr. Corp. Warton, Rept.* no. Ae 225 (see also *Aero. Res. Counc. Lond.* no. 27, 644).

MONIN, A. S. 1959 *Adv. Geophys.* **6**, 331.

MORKOVIN, M. V. 1962 *Mécanique de la Turbulence,* Colloques Intern. du C.N.R.S. no. 108,367.

NEWMAN, B. G. 1951 *Adv. Comm. Aero., Australia, Rept.* no. 53.

PASQUILL, F. 1966 *Quart. J. Roy. Met. Soc.* **92**, 185.

RALSTON, A. & WILF, H. S. (eds.) 1960 *Mathematical Methods for Digital Computers.* Wiley.

RICHTMYER, R. D. 1957 *Difference Methods for Initial-Value Problems.* Interscience.

ROSE, W. G. 1966 *J. Fluid Mech.* **25**, 97.

SARNECKI, A. J. 1958 Ph.D. Dissertation, Cambridge University.

SCHUBAUER, G. B. & KLEBANOFF, P. S. 1951 *Nat. Adv. Comm. Aero. Wash., Rept.* no. 1030.

SCHUBAUER, G. B. & SPANGENBERG, W. G. 1960 *J. Fluid Mech.* **8**, 10.

SPALDING, D. B. & CHI, S. W. 1964 *J. Fluid Mech.* **18**, 117.

STAFF OF MATHEMATICS DIVISION, 1961 Nat. Phys. Lab., Modern computing methods. *Nat. Phys. Lab. Note on Appl. Sci.* 16.

THOMPSON, B. G. J. 1964 *Aero. Res. Counc. Lond.* no. 26,109.

TOWNSEND, A. A. 1956 *The Structure of Turbulent Shear Flow.* Cambridge University Press.

TOWNSEND, A. A. 1961 *J. Fluid Mech.* **11**, 97.

TOWNSEND, A. A. 1965 *J. Fluid Mech.* **22**, 799; **23**, 767.

WINTER, K. G., SMITH, K. G. & GAUDET, L. 1965 *Agardograph,* **97**, 97.